# Experimental Identification of the Generalised Four-Element Transfer Matrix for the Pulsating Gas Installation Element Part I-Theory

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# EXPERIMENTAL IDENTIFICATION OF THE GENERALISED FOUR-ELEMENT TRANSFER MATRIX FOR THE PULSATING GAS INSTALLATION ELEMENT PART I -THEORY

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## Abstract

Most common method used for the compressor installation analysis is the Helmholtz method. This method is based on the assumption that the element is simplified and treated as a straight pipe.

In this paper a theoretical basis for the experimental identification method is shown. The identification is based on the new formulation of a transfer matrix. The pressure to pressure transmission matrix is used. The method requires measurements of pressure pulsations only.

The theory allows calculate pressure waves travelling forward and backward through the identified element. Having pressure wave disassembled into two parts (forward and backward) before and after the installation element the damping, phase change and reflection on the element may be calculated. Thus the transmission matrix has four non-dependent elements.

## Nomenclature

Scalar values

- $b$ - flow damping coefficient
- $c$ – sound velocity,
- $m$, $\dot{m}$ – mass flow rate,
- $l$ – length,
- $p$ – pressure,
- $S$ – cross section area,
- $w$ – velocity,
- $\dot{V}$ – volumetric flow rate,
- $D$ – diameter.

Greek letters

- $\rho$ – density,
- $\tau$ – time,
- $\omega$ – frequency.

Complex values

- $j = \sqrt{-1}$ – imaginary unit,
- P – complex pressure,
- M – complex mass flow rate,

Complex matrices

- $\mathbf{A} = \{a_{ij}\}$ – four pole matrix,
- $\mathbf{Z} = \{z_{ij}\}$ – impedance matrix,
- $\mathbf{T} = \{t_{ij}\}$ – transmittance matrix,

## 1. INTRODUCTION

Pressure pulsation in positive-displacement gas compressor manifolds can significantly affect the quantity of the energy required for gas compression. These effects can be either favourable (by dynamic pressure charging) or negative. They also cause mechanical vibrations of the manifold, noise; they affect the operation of working valves and forced convection in compressor heat exchangers.

This is why the analysis of pressure pulsations is important for both the designer of compression manifolds and the users who introduce modifications.

One of more difficult problems with this analysis is proper description of the acoustic effect of gas manifolds. The analysis is based on the classical Helmholtz model in which an element of the piping system is substituted by composition of straight segments of the pipeline of known length and diameter. The four-pole matrix created in this way has transmission properties: i.e. whole network can be modelled by coupling such elements in series by simple multiplication of their matrices.

Quite a large number of elements of a piping system can be described properly in this way. Unfortunately, not all. For example, oil separators, valves or dampers of special design for which such generalisation is unacceptable because the model is too simplified.

The calculation and simulation methods from CFD group, on the other hand, are not convenient for these applications although they are found very useful for the analysis of motor vehicle dampers. This is because the piping system is complex and vast, in some cases they can be hundred of meters long. There fore modelling it as whole by CFD method is time consuming, both for computer, but also what is more important for a programmer who has to feed all the geometry to the computer.

The aim of the present project was to work out a method that would allow the experimental identification of any element of a gas manifold after the generalised Helmholtz model. The matrix identified by this method, describing the element of installation, would have the features of transmission matrix but would not be based on a simplified model of a pipeline segment.

## 2. THEORETICAL BASIS OF THE METHOD

The classical Helmholtz model is based on the solution of wave equation for a straight piping section (1). The result is a four-pole matrix as shown in (2), (3). The elements of this matrix $\{a_{ij}\}$ are calculated for a straight pipe segment. The complex impedance **Z** with elements $\{z_{ij}\}$ defined by (3) is used alongside the four-pole matrix. It may be easily transformed both ways **A-Z** and **Z-A**.

$$\begin{cases} \frac{\partial p}{\partial x} = \frac{1}{S}\frac{\partial \dot{m}}{\partial \tau} + \frac{b}{S}\cdot \dot{m} \\ \frac{\partial \dot{m}}{\partial x} = \frac{S}{c^2}\frac{\partial p}{\partial \tau} \end{cases} \qquad (1)$$

$$\begin{bmatrix} P_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} ch\gamma L & Z_f sh\gamma L \\ \frac{1}{Z_f} sh\gamma L & ch\gamma L \end{bmatrix} \cdot \begin{bmatrix} P_2 \\ M_2 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} P_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} P_2 \\ M_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \cdot \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \tag{3}$$

where:

$$\gamma = \sqrt{(b + j\omega)\left(\frac{j\omega}{c^2}\right)}, \qquad Z_f = \frac{1}{S}\sqrt{\frac{(b + j\omega)}{\frac{j\omega}{c^2}}} \tag{4}$$

In order to work out experimental methods of thermodynamic identification of gas manifold elements the following assumptions were adopted:

a) The identified object can be described mathematically in the same general shape as the Helmholtz model presented above, which would lead to generalisation of it.
b) During experiment only interaction between one inlet and one outlet is investigated, assuming that interaction between several branches can be covered by the principle of superposition.
c) During experiment only pressure pulsation at any chosen gas manifold points, before and behind the element, are measured. Measurement of pulsating mass flow rate cannot at present be performed in an operating manifold for technical reasons.

The element of a manifold chosen for investigation is between cross-sections $i$+1 and $i$+2 in Fig. 2.1.

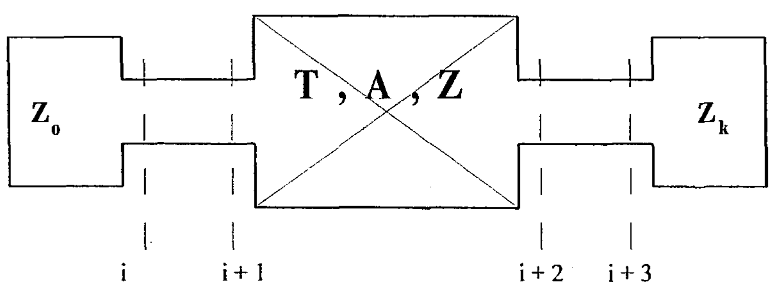


Figure 2.1. A schematic model of a manifold element for identification purpose, (between the cross-sections i+1 and i+2, straight pipe i-i+1 and i+2-i+3).

The pipeline segment, which can be treated as a straight pipe of wave impedance $Z_i$, connects the analysed element with the source of excitation, and thus the entrance of the manifold with the positive-displacement compressor. The other outlet from the tested element ($i$+2) is connected with closing impedance $Z_k$ by a straight pipeline of wave impedance $Z_{i+2}$. Both closing $Z_k$ and entrance of manifold $Z_0$ connected to the compressor can consist of many various branches and devices but this does not affect the results of the presented investigation.

As mentioned before, pressure pulsations measured at any points $i$ - $i$+3 are composed of progressive wave and reflective one. The number of these reflections is infinite because acoustic waves are reflected from both the closing and entrance of the gas manifold. In certain conditions these reflections can intensify or attenuate each other. Here the acoustic wave was divided only into progressive - denoted $p^+$ and reflected - denoted $p^-$ without defining the multiplicity of its components reflection. In this way, between any two points (e.g. $i$+1, $i$+2) the following dependencies in the complex domain can be written:

$$\begin{bmatrix} P_{i+1}^+ \\ P_{i+1}^- \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} P_{i+2}^+ \\ P_{i+2}^- \end{bmatrix} \tag{5}$$

Matrix **T** will be called the matrix of pressure transmittance. A question arises, if it exists what is the dependence between **T A** and **Z** matrices? It shall be noticed here that the transmittance matrix has transmission properties:

$$\begin{bmatrix} P_i^+ \\ P_i^- \end{bmatrix} = [\mathbf{T}_i][\mathbf{T}_{i+1}][\mathbf{T}_{i+2}] \cdot \begin{bmatrix} P_{i+3}^+ \\ P_{i+3}^- \end{bmatrix} \tag{6}$$

This dependence is valid for any series connection of elements of matrix $[\mathbf{T}_k]$ on condition that these matrices exist (e.g. a gap in the gas manifold does not have transmittance matrix because the pressure pulsations on one side are completely independent of the pulsations on the other side). A dividing element can be heavy damping or a large tank, which can be treated as full opening.

Let us consider the investigated element for which matrix **T** existence is assured. Let us assume, after the model in Fig. 3.1, that the analysed element is connected with the whole gas manifold by a straight pipeline of wave impedance $Z_{fi}$, and a pipeline of impedance $Z_{fi+2}$ on the outlet. The general solution of equation of acoustic wave propagation in connecting pipelines is of the same shape as equation (2) presented above.

$$\begin{cases} P_i = P_i^+ + P_i^- = A_1 e^{-\gamma x} + A_2 e^{\gamma x} \\ M_i = M_i^+ + M_i^- = \dfrac{A_1}{Z_{fi}} e^{-\gamma x} - \dfrac{A_2}{Z_{fi}} e^{\gamma x} \end{cases} \tag{7}$$

Four-pole matrix for the identified element is as follows:

$$\begin{cases} P_{i+1} = a_{11} P_{i+2} + a_{12} M_{i+2} \\ M_{i+1} = a_{21} P_{i+2} + a_{22} M_{i+2} \end{cases} \tag{8}$$

Equation (8) may be also written down for point $i$+1:

$$\begin{cases} P_{i+1} = P_{i+1}^+ + P_{i+1}^- \\ M_{i+1} = \dfrac{1}{Z_{fi}} \left( P_{i+1}^+ - P_{i+1}^- \right) \end{cases} \tag{9}$$

and for point $i$+2:

$$\begin{cases} P_{i+2} = P_{i+2}^+ + P_{i+2}^- \\ M_{i+2} = \dfrac{1}{Z_{fi+2}} \left( P_{i+2}^+ - P_{i+2}^- \right) \end{cases} \tag{10}$$

Substituting formulas (9) and (10) in the equation (8) for $M_{i+1}$, $M_{i+2}$ after some rearrangements we obtain:

$$\begin{cases} P_{i+1}^+ = \dfrac{1}{2} \left\{ a_{11} + \dfrac{a_{12}}{Z_{fi+2}} + a_{21} Z_{fi} + a_{22} \dfrac{Z_{fi}}{z_{fi+2}} \right\} P_{i+2}^+ + \\ + \dfrac{1}{2} \left\{ a_{11} - \dfrac{a_{12}}{Z_{fi+2}} + a_{21} Z_{fi} - a_{22} \dfrac{Z_{fi}}{Z_{fi+2}} \right\} P_{i+2}^- \\ P_{i+1}^- = \dfrac{1}{2} \left\{ a_{11} + \dfrac{a_{12}}{Z_{fi+2}} - a_{21} Z_{fi} - a_{22} \dfrac{Z_{fi}}{Z_{fi+2}} \right\} P_{i+2}^+ + \\ + \dfrac{1}{2} \left\{ a_{11} - \dfrac{a_{12}}{Z_{fi+2}} - a_{21} Z_{fi} + a_{22} \dfrac{Z_{fi}}{Z_{fi+2}} \right\} P_{i+2}^- \end{cases} \tag{11}$$

Finally for the identified element:

$$\left.\begin{aligned}
t_{11} &= \frac{1}{2}\left( a_{11} + \frac{a_{12}}{Z_{fi+2}} + a_{21} Z_{f1} + a_{22} \frac{Z_{fi}}{Z_{fi+2}} \right) \\
t_{12} &= \frac{1}{2}\left( a_{11} - \frac{a_{12}}{Z_{fi+2}} + a_{21} Z_{fi} - a_{22} \frac{Z_{fi}}{Z_{fi+2}} \right) \\
t_{21} &= \frac{1}{2}\left( a_{11} + \frac{a_{12}}{Z_{fi+2}} - a_{21} Z_{fi} - a_{22} \frac{Z_{fi}}{Z_{fi+2}} \right) \\
t_{22} &= \frac{1}{2}\left( a_{11} - \frac{a_{12}}{Z_{fi+2}} - a_{21} Z_{fi} + a_{22} \frac{Z_{fi}}{Z_{fi+2}} \right)
\end{aligned}\right\} \tag{12}$$

or :

$$\left.\begin{aligned}
a_{11} &= \frac{1}{2}\left(t_{11} + t_{12} + t_{21} + t_{22}\right) \\
a_{12} &= \frac{1}{2} Z_{fi}\left(t_{11} - t_{12} + t_{21} - t_{22}\right) \\
a_{21} &= \frac{1}{2} \frac{1}{Z_{fi+2}}\left(t_{11} + t_{12} - t_{21} - t_{22}\right) \\
a_{22} &= \frac{1}{2} \frac{Z_{fi+2}}{Z_{fi}}\left(t_{11} - t_{12} - t_{21} + t_{22}\right)
\end{aligned}\right\} \tag{13}$$

Knowing matrix **A**, matrix **Z** for identified element can be easily calculated:

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \frac{1}{a_{21}} \begin{bmatrix} a_{11} & |a| \\ 1 & a_{22} \end{bmatrix} \tag{14}$$

It follows that knowing the pressure transmission matrix and wave impedance of the connecting pipelines; it is possible to define any matrix of the investigated manifold, which means that the identification of the element is unique.

In this way the following stage of investigation was reached:

- basic pressure composite transmission matrix **T** was defined; when creating it measurement of transient mass rate of flow is not necessary;
- it was found that for the analysis of pulsations in a gas manifold it is practically sufficient to know $p^+$ and $p^-$, i.e. pressure waves progressing in opposite directions. A way of determining them will be discussed further on in the paper.

Let us assume now, that it is possible to measure the pressure pulsations in the selected cross-sections of the gas manifold, i.e. $i$, $i$+1, $i$+2, $i$+3. As a result four functions $p_i(\tau)$, $p_{i+1}(\tau)$, $p_{i+2}(\tau)$, $p_{i+3}(\tau)$ are obtained.

Let us consider one of Fourier components for any frequency $\omega$. This does not undermine the generality of this analysis because it will also refer to any other frequency.

Using Fourier composite transform, the following is obtained in the complex domain:

$$
\begin{aligned}
&P_i(\omega) = F(p_i)\\
&P_{i+1}(\omega) = F(p_{i+1})\\
&P_i^+(\omega) = F(p_i^+) \quad P_i^-(\omega) = F(p_i^-)
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&P_i(\omega) = P_i^+(\omega) + P_i^-(\omega)\\
&P_{i+1}(\omega) = P_{i+1}^+(\omega) + P_{i+1}^-(\omega)
\end{aligned}
\tag{16}
$$

It can be assumed that element ($i$, $i$+1) whose acoustic properties should be defined, is attached to the rest of the gas manifold by two elements ($i$-1, $i$) and ($i$+1, $i$+2) for which any of **A**, **T** or **Z** matrices are known. Most often these are connections which are parts of pipelines of diameter $D$ and length $L$. The acoustic velocity $c$ in the pipeline is also known. In case of larger lengths damping coefficient should also be taken into consideration. The impedance matrix **Z** of such an element has the form (3).

Matrix **T** is calculated from dependence (12). Using these last formula equations to define $P_i^+$, $P_i^-$, $P_{i+1}^+$, $P_{i+1}^-$ can be written down.

$$
\left.
\begin{aligned}
&P_i = P_i^+ + P_i^-\\
&P_{i+1} = P_{i+1}^+ + P_{i+1}^-\\
&P_i^+ = t_{11} P_{i+1}^+ + t_{12} P_{i+1}^-\\
&P_i^- = t_{21} P_{i+1}^+ + t_{22} P_{i+1}^-
\end{aligned}
\right\}
\tag{17}
$$

The set of equations (17) is an algebraic linear set of four equations with four unknowns. It should be solved in amplitude-frequency complex numbers. The solution of the set is:

$$
\left.
\begin{aligned}
P_{i+1}^{+} &= \frac{P_i - P_{i+1}(t_{12}+t_{22})}{(t_{11}+t_{21})-(t_{12}+t_{22})} \\
P_{i+1}^{-} &= \frac{P_i - P_{i+1}(t_{11}+t_{21})}{(t_{11}+t_{22})-(t_{11}+t_{21})} \\
P_i^{+} &= \frac{t_{11}\cdot t_{22} - t_{21}\cdot t_{12}}{t_{12}+t_{22}} \cdot P_{i+1}^{+} + \frac{t_{12}}{t_{12}+t_{22}} \cdot P_i \\
P_i^{-} &= \frac{t_{11}\cdot t_{22} - t_{12}\cdot t_{21}}{t_{21}+t_{11}} \cdot P_{i+1}^{-} + \frac{t_{21}}{t_{21}+t_{11}} \cdot P_i
\end{aligned}
\right\} \tag{18}
$$

Dependencies (18) allow us to separate the progressing and returning waves of pressure pulsation in a volumetric compressor manifold. To do this it is necessary to measure the pressure pulsations at two points of the manifold separated by elements of known matrix **T**.

For the investigated element, placed between points $(i+1)$ and $(i+2)$, the values of the separated pressure $P^+$ and $P^-$ at both points are determined in this way.

In some cases, when mean velocity of the medium is significant and can affect the velocity of acoustic wave propagation, a correction for medium mean velocity should be taken into account in calculations.

## 3.CONCLUSIONS

The conclusion of this paper is that it is possible to identify experimentally four elements of the matrix describing the acoustic reaction of any element of the manifold, only on the basis of measured pressure pulsation curves at selected points of the manifold. Basis for this is the division of the pressure waves into going forwards and backwards waves.

Other conclusions will be given in part II.

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